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Incompleteness and closure of a linear span of exponential system in a weighted Banach space ☆

G.T. Deng

Department of Mathematics, Beijing Normal University, 100875 Beijing, People's Republic of China

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Abstract

In this paper, we obtain a necessary and sufficient condition for the incompleteness of complex exponential polynomials in C_α, where C_α is a weighted Banach space of complex continuous functions f on the real axis $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, in the uniform norm with respect to the weight $\alpha(t)$. We also prove that, if the above condition of incompleteness holds, then each function in the closure of complex exponential polynomials can be extended to an entire function represented by a Dirichlet series.

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1. Introduction

In the study of weighted approximation on the real line $\mathbb{R}$, we start with a nonnegative continuous function $\alpha(t)$, henceforth called a weight, defined on $\mathbb{R}$. We usually suppose that

$$\lim_{|t|\to\infty} |t|^{-1}\alpha(t) = \infty. \tag{1}$$

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E-mail address: denggt@bnu.edu.cn.

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Given a weight $\alpha(t)$, we take a weighted Banach space C_α consisting of complex continuous functions $f(t)$ defined on $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, and write

$$||f||_\alpha = \sup\{|f(t)e^{-\alpha(t)}| : t \in \mathbb{R}\}$$

for $f \in C_\alpha$. Denote by $M(\Lambda)$ the set of complex exponential polynomials which are finite linear combinations of the exponential system $\{e^{\lambda t} : \lambda \in \Lambda\}$, where $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ is a sequence of complex numbers in the open right half-plane $\mathbb{C}_+ = \{z = x + iy : x > 0\}$. Our condition (1) guarantees that $M(\Lambda)$ is a subspace of C_α, we then ask whether $M(\Lambda)$ is incomplete [6] in C_α in the norm $|| \; ||_\alpha$—this is the so-called weighted exponential polynomial approximation, which is similar to the classical Bernstein problem on weighted polynomial approximation [3].

Motivated by Bernstein's problem and Malliavin's Method [3], we find a necessary and sufficient condition for $M(\Lambda)$ to be incomplete in C_α. Inspired by Borwein's and Erdélyi's results [2, p. 311], we also prove that, if the above condition of incompleteness holds, then the closure of $M(\Lambda)$ in C_α is the set of all $f \in C_\alpha$ which can be extended to an entire function represented by a Dirichlet series.

Our main conclusions are as follows:

Theorem 1. *Let $\alpha(t)$ be a nonnegative convex function on $\mathbb{R}$ satisfying* (1). *Let $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ be a sequence of complex numbers in $\mathbb{C}_+$ satisfying*

$$\Theta(\Lambda) = \sup\{|\theta_n| : n = 1, 2, \ldots\} < \frac{\pi}{2} \tag{2}$$

and

$$\delta(\Lambda) = \inf\{|\lambda_{n+1}| - |\lambda_n| : n = 1, 2, \ldots\} > 0. \tag{3}$$

Then $M(\Lambda)$ is incomplete in C_α if and only if there exists $a \in \mathbb{R}$ such that

$$\int_1^{+\infty} \frac{\alpha(\lambda(t) - a)}{1 + t^2}\, dt < \infty, \tag{4}$$

where

$$\lambda(r) = \begin{cases} 2 \sum\limits_{|\lambda_n| \leqslant r} \dfrac{\cos\theta_n}{|\lambda_n|} & \text{if } r \geqslant |\lambda_1|, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 2. *Let $\alpha(x)$ be a nonnegative convex function on $\mathbb{R}$ satisfying* (1). *Let Λ be a sequence of complex numbers in $\mathbb{C}_+$ satisfying* (2) *and* (3). *Suppose that $\lambda(r)$ is unbounded on $(0, \infty)$, where $\lambda(r)$ is defined by* (5). *If $M(\Lambda)$ is incomplete in C_α, then each function f in the closure of $M(\Lambda)$ can be extended to an entire function $g(z)$*

represented by a Dirichlet series

$$g(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}. \tag{6}$$

Remark 1. The case $\Theta(\Lambda) = 0$ and $\delta(\Lambda) > 0$ in Theorem 1 is obtained by Malliavin [4].

Borwein and Erdélyi [2] have proved a similar result to Theorem 2 for the closure of M*ü*ntz polynomials on a finite interval when $\Theta(\Lambda) = 0$, $\delta(\Lambda) > 0$ and $\lambda(r)$ is bounded on $(0, \infty)$. Thus Theorem 2 is a further generalization of some results in [2].

2. Proof of theorems

In order to prove Theorems 1 and 2, we need the following technical lemmas.

Lemma 1 (Malliavin [4]). *Let $\beta(t)$ be a nonnegative convex function on $\mathbb{R}$ satisfying* (1), *and assume that*

$$\beta^*(t) = \sup\{xt - \beta(x) : x \in \mathbb{R}\}, \quad t \in \mathbb{R} \tag{7}$$

is the Young transform [6] *of the function $\beta(x)$. Suppose that $\lambda(r)$ is an increasing function on $[0, \infty)$ satisfying*

$$\lambda(R) - \lambda(r) \leqslant A(\log R - \log r + 1) \ (R > r > 1). \tag{8}$$

Then there exists an analytic function $f(z) \not\equiv 0$ in $\mathbb{C}_+$ satisfying

$$|f(z)| \leqslant A \exp\{Ax + \beta(x) - x\lambda(|z|)\}, \quad z = x + iy \in \mathbb{C}_+, \tag{9}$$

if and only if there exists $a \in \mathbb{R}$ such that

$$\int_1^{+\infty} \frac{\beta^*(k(t) - a)}{1 + t^2}\, dt < \infty. \tag{10}$$

Remark 2. Lemma 1 is a result of Malliavin's uniqueness theorem [4] about Watson's problem. (Hereafter, we denote a positive constant by A, not necessarily the same at each occurrence.)

Lemma 2. *If Λ is a sequence of complex numbers in $\mathbb{C}_+$ satisfying* (2) *and* (3), *then the function*

$$G(z) = \prod_{n=1}^{\infty} \left(\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\bar{\lambda}_n}} \right) \exp\left(\frac{z}{\lambda_n} + \frac{z}{\bar{\lambda}_n} \right). \tag{11}$$

is analytic in the closed right half-plane $\bar{\mathbb{C}}_+ = \{z = x + iy : x \geqslant 0\}$, and satisfies the following inequalities:

$$|G(z)| \leqslant \exp\{x\lambda(r) + Ax\}, \quad z \in \mathbb{C}_+, \tag{12}$$

$$|G(z)| \geqslant \exp\{x\lambda(r) - Ax\}, \quad z \in C(\Lambda, \delta_0), \tag{13}$$

$$|G'(\lambda_n)| \geqslant \exp\{\mathrm{Re}\,\lambda_n \lambda(|\lambda_n|) - A\mathrm{Re}\,\lambda_n\}, \quad n = 1, 2, \dots, \tag{14}$$

where $r = |z|, 4\delta_0 = \delta(\Lambda)$ *and* $C(\Lambda, \delta_0) = \{z \in \mathbb{C}_+ : |z - \lambda_n| \geqslant \delta_0, \ n = 1, 2, \dots\}$.

Proof of Lemma 2. We will use a similar method to that of the proof of Fuch's lemma in [1]. Let

$$e_n(z) = \left|\frac{z - \lambda_n}{z + \bar{\lambda}_n}\right|^2 = 1 - \frac{4x|\lambda_n|\cos\theta_n}{|z + \bar{\lambda}_n|^2}, \qquad E_n(z) = 2x\frac{\cos\theta_n}{|\lambda_n|} + \frac{1}{2}\log e_n(z),$$

where $x = r\cos\theta > 0$. Since $\log(1 - t) \leqslant -t$ for $t \in [0, 1)$, we have

$$E_n(z) \leqslant \frac{2xr(2|\lambda_n|\cos\theta\cos\theta_n + r)\cos\theta_n}{|\lambda_n|\,|z + \bar{\lambda}_n|^2}, \quad z \in \mathbb{C}_+. \tag{15}$$

Moreover, since $\log(1 - t) \geqslant -t - \frac{\delta t^2}{1+\delta}, \ t \in [0, (1 + \delta)^{-1}]$ for $\delta > 0$, we also have

$$E_n(z) \geqslant -\frac{(1 + \delta)|\lambda_n|^2 x^2 \cos^2\theta_n}{4\delta|z + \bar{\lambda}_n|^4} \tag{16}$$

for $z \in \{z \in \mathbb{C}_+ : \ 0 \leqslant 1 - e_n(z) \leqslant (1 + \delta)^{-1}\}$. Let $\eta = \eta(\delta) = 2\delta + 2\sqrt{\delta(1 + \delta)}, \ \delta > 0$. If $|\lambda_n| \geqslant (1 + \eta)r, \ z \in \mathbb{C}_+$ and $r = |z|$, then

$$0 \leqslant 1 - e_n(z) \leqslant (1 + \delta)^{-1},$$

and thus

$$|E_n(z)| \leqslant 2xr\frac{\cos\theta_n}{|\lambda_n|^2}\left(\frac{3}{\eta^4} + 2\frac{1 + \delta}{\eta^6\delta}\right). \tag{17}$$

By (3), $\sum_{n=1}^{\infty} |\lambda_n|^{-2}$ converge. Thus $G(z)$ is the quotient of convergent canonical products. As a consequence, it is analytic in $\{z \in \mathbb{C} : z \neq -\bar{\lambda}_n, \ n = 1, 2, \dots\}$. To prove (12)–(14), write $G(z) = \Pi_1\Pi_2$, where Π_1 contains the terms with $|\lambda_n| \leqslant (1 + \eta)r$, $r = |z|, \ \eta = \eta(\delta) = 2\delta + 2\sqrt{\delta(1 + \delta)}$. Apply (15) to the factors in Π_1 and apply (17) to those in Π_2. Then, for $x \geqslant 0$, we obtain

$$\log|\Pi_1| \leqslant 2x \sum_{|\lambda_n| \leqslant (1+\eta)r} \frac{\cos\theta_n}{|\lambda_n|} \leqslant x\lambda(r) + Ax \tag{18}$$

and

$$|\log|\Pi_2|| \leqslant Axr \sum_{|\lambda_n| > (1+\eta)r} \frac{\cos\theta_n}{|\lambda_n|^2} \leqslant Ax, \tag{19}$$

where we have used the inequality $|\lambda_n| \geqslant An$ in the last part of (18) and (19). Therefore (12) holds.

In order to prove (13), we may use (19) again for Π_2. For Π_1, let $N(r)$ denote the number of λ_n satisfying $|\lambda_n| \leqslant r = |z|$, then $|N(R) - N(r)| \leqslant A|R - r| + A (R > r > 1)$.

We consider the following two cases for Π_1 : (i) $z \in \{z = re^{i\theta} \in C(\Lambda, \delta_0) : \Theta(\Lambda) + 2\varepsilon_1 \leqslant |\theta| < \frac{\pi}{2}\}$ and (ii) $z \in \{z = re^{i\theta} \in C(\Lambda, \delta_0) : |\theta| < \Theta(\Lambda) + 2\varepsilon_1\}$, where $4\varepsilon_1 = (\frac{\pi}{2} - \Theta(\Lambda))$. In case (i), let $\delta_1 = \sin^2 \varepsilon_1$, then

$$|z + \bar{\lambda}_n|^2 \geqslant 2r|\lambda_n|\delta_1, \;\; 0 < 1 - e_n(z) \leqslant (1 + \delta_1)^{-1},$$

consequently

$$\begin{aligned}\log|\Pi_1| &\geqslant 2x \sum_{|\lambda_n| \leqslant (1+\eta)r} \frac{\cos\theta_n}{|\lambda_n|} - 2x \frac{1+\delta_1}{\delta_1} \sum_{|\lambda_n| \leqslant (1+\eta)r} \frac{|\lambda_n| \cos\theta_n}{|z + \bar{\lambda}_n|^2} \\ &\geqslant -Ax + x\lambda(r).\end{aligned}$$

This implies that (13) holds in this case. In case (ii), (2) and Stirling's formula give

$$\prod_1^N |\lambda_n - z| \geqslant \delta_0^N N(r)!(N - N(r))! \geqslant \left(\frac{N}{A}\right)^N,$$

$$\prod_1^N |\lambda_n + z| \leqslant (Ar)^N.$$

Thus

$$\log|\Pi_1| \geqslant N(\log N - \log(Ax)) + x\lambda(r) \geqslant x\lambda(r) - Ax - A,$$

where $N = N((1 + \eta)r)$, $r = |z|$ and in the last inequality, we use $N(\log N - \log a) \geqslant -ae^{-1}$ for $a > 0$. Therefore (13) is proved. Eq. (14) can be proved by the same method. □

Proof of Theorem 1. If the space $M(\Lambda)$ is incomplete in C_α, then there exists a bounded linear functional T such that $||T|| = 1$ and $T(e^{\lambda t}) = 0$ for $\lambda \in \Lambda$. So by the Riesz representation theorem, there exists a complex measure μ satisfying

$$||\mu|| = \int_{-\infty}^{+\infty} e^{\alpha(t)} d|\mu(t)| = ||T||$$

and

$$T(h) = \int_{-\infty}^{+\infty} h(t)\, d\mu(t), \;\; h \in C_\alpha.$$

Therefore the function

$$f(z) = \frac{1}{G(z)} \int_{-\infty}^{+\infty} e^{tz} d\mu(t)$$

is analytic in the open right half-plane $\mathbb{C}_+$ and continuous in the closed right half-plane $\bar{\mathbb{C}}_+ = \{z = x + iy : x \geqslant 0\}$, where $G(z)$ is defined by (11). By Lemma 2, we obtain

$$|f(z)| \leqslant \exp\{\alpha^*(x) - x\lambda(|z|) + Ax\},$$

where

$$\alpha^*(x) = \sup\{xt - \alpha(t) : t \in \mathbb{R}\} \tag{20}$$

is the Young transform [5] of the convex function $\alpha(x)$. We may assume, without loss of generality, that $\alpha(0) = 0$. As is known [5], $\alpha^*(x)$ is a convex nonnegative function which also satisfies $\alpha^*(0) = 0$ and $(\alpha^*)^* = \alpha$. Since $\alpha^*(x) \geqslant xt - \alpha(x)$ for $x > 0, t > 0$ and $\alpha^*(x) \geqslant |x||t| - \alpha(x)$ for $x < 0, t < 0$, so (1) is also satisfied. Hence for $t > 0$,

$$\sup\{xt - \alpha^*(x) : x \geqslant 0\} = \alpha(t).$$

Since the condition $\delta(\Lambda) > 0$ implies that (8) holds, we see from Lemma 1 with $\beta = \alpha^*$ that (4) holds. This completes the proof of necessity of Theorem 1.

Next, we turn to the proof of sufficiency of Theorem 1. Assume that there exists a real number a such that the integral

$$A_1 = \frac{8}{\pi}\int_0^{\infty} \frac{\alpha(\lambda(t) - a)}{1 + t^2}\, dt < \infty. \tag{21}$$

Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\lambda(t) - a)$ for $t \geqslant 0$ and let $u(z)$ be the Poisson integral of $\varphi(t)$, i.e.,

$$u(x + iy) = \frac{x}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{x^2 + (y - t)^2}\, dt.$$

Then $u(x + iy)$ is harmonic in the half-plane $\mathbb{C}_+$ and there exists an analytic function $g_1(z)$ on $\mathbb{C}_+$ satisfying

$$A_1 x \geqslant \mathrm{Re}\, g_1(z) = 4u(z) \geqslant (x - 1)(\lambda(r) - a) - \alpha^*(x - 1),$$

where $z = x + iy$, $r = |z|, x > 1$. Let

$$g_0(z) = \frac{G(z)}{(1 + z)^N} \exp\{-g_1(z) - Nz - N\},$$

where N is a large positive integer and $G(z)$ is defined by (11). By (12), we have

$$|g_0(z)| \leqslant \frac{1}{1 + |z|^2} \exp\{\alpha^*(x - 1) - x\}, \quad z \in \mathbb{C}_+. \tag{22}$$

Let

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(1 + iy) e^{-(1+iy)t}\, dy. \tag{23}$$

Then $h_0(t)$ is continuous on $(-\infty, +\infty)$. By Cauchy's formula,

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(x + iy) e^{-(x+iy)t}\, dy, \quad x > 0. \tag{24}$$

We obtain from (22), (24) and the formula of the Young transform $(\alpha^*)^* = \alpha$ that

$$|h_0(t)| \leqslant \exp(-\alpha(t) - |t|) \tag{25}$$

and

$$g_0(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) e^{tz}\, dt, \quad x > 0. \tag{26}$$

Therefore the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) h(t)\, dt, \quad h \in C_\alpha, \tag{27}$$

satisfies $T(e^{\lambda t}) = 0$ for $\lambda \in \Lambda$, and

$$||T|| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |h_0(t)| e^{\alpha(t)} dt > 0.$$

By the Riesz representation theorem, the space $M(\Lambda)$ is incomplete in C_α. This completes the proof of Theorem 1. □

Proof of Theorem 2. If $M(\Lambda)$ is incomplete in C_α, then it follows from Theorem 1 that there exists a real number a such that (21) holds. The proof of necessity of Theorem 1 implies that there exist an analytic function $g_0(z)$ on $\mathbb{C}_+$ and a continuous function $h_0(t)$ on $\mathbb{R}$ such that g_0 and h_0 satisfy (22)–(27).

Let $\varphi_n(z) = (z - \lambda_n)^{-1} g_0(z), \varphi_n(\lambda_n) = g_0'(\lambda_n),\ n = 1, 2, \ldots$, and let

$$h_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_n(1 + iy) e^{-(1+iy)t}\, dy.$$

By a proof similar to that of necessity of Theorem 1, we get that $h_n(t)$ is continuous on $\mathbb{R}$, and satisfies

$$|h_n(t)| \leqslant \exp(-\alpha(t) - |t|), \tag{28}$$

and that dual relation

$$h_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_n(x + iy) e^{-(x+iy)t}\, dy \quad (x > 0)$$

and

$$\varphi_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_n(t) e^{tz}\, dt \quad (x > 0)$$

holds. Note that $0 \leqslant \operatorname{Re} g_1(z) \leqslant A_1 x$ for $x \geqslant 1$. We obtain, by (14),

$$|\varphi_n(\lambda_n)| \geqslant \exp\{\operatorname{Re} \lambda_n \lambda(|\lambda_n|) - A \operatorname{Re} \lambda_n - A\}, \quad n = 1, 2, \ldots .$$

By (28), there exists a constant A_2 independent of Λ such that the function

$$\psi_n(t) = \frac{1}{\sqrt{2\pi}} \frac{h_n(t)}{\varphi_n(\lambda_n)}$$

satisfies

$$|\psi_n(t)e^{\alpha(t)}| \leqslant \exp\{A_2 + A_2 \operatorname{Re}\lambda_n - \operatorname{Re}\lambda_n \lambda(\lambda_n) - |t|\}, \tag{29}$$

and

$$\int_{-\infty}^{+\infty} \psi_n(t)e^{\lambda_k t}dt = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Define a linear functional T_n on $M(\Lambda)$ by

$$T_n(P) = a_n = \int_{-\infty}^{+\infty} \sum a_k \psi_n(t) e^{\lambda_k t}\, dt$$

for each exponential polynomial $P(t) = \sum a_k e^{\lambda_k t} \in M(\Lambda)$. By (29) and (30),

$$|T_n(P)| \leqslant 2||P||_\alpha \exp\{A_2 + A_2 \operatorname{Re}\lambda_n - \operatorname{Re}\lambda_n \lambda(|\lambda_n|)\}.$$

Hence T_n is a bounded linear functional on $M(\Lambda)$ and may be extended to a bounded linear functional (denoted by $\bar{T}_n$) on C_α by the Hahn–Banach theorem with

$$||\bar{T}_n|| = ||T_n|| \leqslant C_n = 2\exp\{A_2 + A_2 \operatorname{Re}\lambda_n - \operatorname{Re}\lambda_n \lambda(|\lambda_n|)\}. \tag{31}$$

If f belongs to the closure of $M(\Lambda)$ in C_α, then there exists a sequence of exponential polynomials

$$P_k(t) = \sum_{n=1}^{k} a_{nk} \exp(\lambda_n t) \in M(\Lambda)$$

such that

$$||f - P_k||_\alpha \to 0 \quad \text{as } k \to \infty.$$

Since $\lambda(r)$ is unbounded on $(0, \infty)$, by (30), the function

$$g(z) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n z)$$

is an entire function, where $a_n = \bar{T}_n(f),\ n = 1, 2, \ldots$. Note that

$$|a_n - a_{nk}| = |\bar{T}_n(f) - \bar{T}_n(P_k)| \leqslant C_n ||f - P_k||_\alpha, \quad n = 1, 2, \ldots .$$

We obtain that, for $x \in \mathbb{R}$,

$$|f(x) - g(x)| \leqslant |f(x) - P_k(x)| + |P_k(x) - g(x)|$$
$$\leqslant e^{\alpha(x)}||f - P_k||_\alpha + \sum_{n=1}^{k} |a_{nk} - a_n| e^{\operatorname{Re}\lambda_n x} + \sum_{n=k+1}^{\infty} |a_n| e^{\operatorname{Re}\lambda_n x}.$$

Letting $k \to \infty$, we obtain that $f(x) = g(x)$ for $x \in \mathbb{R}$. This completes the proof of Theorem 2. □

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References

[1] R.P. Boas Jr., Entire Functions, Academic Press, New York, 1954.
[2] P.B. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, NY, 1995.
[3] P. Koosis, The Logarithmic Integral, Vol. I, Cambridge University Press, Cambridge, 1988.
[4] P. Malliavin, Sur quelques procédés d'extrapolation, Acta Math. 83 (1955) 179–255.
[5] R. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[6] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.